

# Solving inequalities

Inequalities are mathematical expressions involving the symbols  $>$ ,  $<$ ,  $\geq$  and  $\leq$ . To 'solve' an inequality means to find a range, or ranges, of values that an unknown  $x$  can take and still satisfy the inequality.

In this unit inequalities are solved by using algebra and by using graphs.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- solve simple inequalities using algebra
- solve simple inequalities by drawing graphs
- solve inequalities in which there is a modulus symbol
- solve quadratic inequalities

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# 1. Introduction

The expression  $5x - 4 > 2x + 3$  looks like an equation but with the equals sign replaced by an arrowhead. It is an example of an **inequality**.

This denotes that the part on the left,  $5x - 4$ , is greater than the part on the right,  $2x + 3$ . We will be interested in finding the values of  $x$  for which the inequality is true.

We use four symbols to denote inequalities:



## Key Point

- $>$  is greater than
- $\geq$  is greater than or equal to
- $<$  is less than
- $\leq$  is less than or equal to

Notice that the arrowhead always points to the smaller expression.

# 2. Manipulation of inequalities

Inequalities can be manipulated like equations and follow very similar rules, but there is one important exception.

If you add the same number to both sides of an inequality, the inequality remains true.

If you subtract the same number from both sides of the inequality, the inequality remains true.

If you multiply or divide both sides of an inequality by the same **positive** number, the inequality remains true.

But if you multiply or divide both sides of an inequality by a negative number, the inequality is no longer true. In fact, the inequality becomes reversed. This is quite easy to see because we can write that  $4 > 2$ . But if we multiply both sides of this inequality by  $-1$ , we get  $-4 > -2$ , which is not true. We have to reverse the inequality, giving  $-4 < -2$  in order for it to be true.

This leads to difficulties when dealing with variables, because a variable can be either positive or negative. Consider the inequality

$$x^2 > x$$

It looks as though we might be able to divide both sides by  $x$  to give

$$x > 1$$

But, in fact, we cannot do this. The two inequalities  $x^2 > x$  and  $x > 1$  are not the same. This is because in the inequality  $x > 1$ ,  $x$  is clearly greater than 1. But in the inequality  $x^2 > x$  we have to take into account the possibility that  $x$  is negative, since if  $x$  is negative,  $x^2$  (which must be positive or zero) is always greater than  $x$ . In fact the complete solution of this inequality is  $x > 1$  or  $x < 0$ . The second part of the solution must be true since if  $x$  is negative,  $x^2$  is always greater than  $x$ . We will see in this unit how inequalities like this are solved. Great care has to be taken when solving inequalities to make sure you do not multiply or divide by a negative number by accident. For example

saying that  $x > y$ , implies that  $x^2 > y^2$  only if  $x$  and  $y$  are positive.

We can see the necessity of the condition that both  $x$  and  $y$  are positive by considering  $x = 1$  and  $y = -10$ . Since  $x$  is positive and  $y$  negative it follows that  $x > y$ ; but  $x^2 = 1$  and  $y^2 = 100$  and so  $y^2 > x^2$ .



### Key Point

When solving an inequality:

- you can add the same quantity to each side
- you can subtract the same quantity from each side
- you can multiply or divide each side by the same **positive** quantity

If you multiply or divide each side by a negative quantity, the inequality symbol must be reversed.

## 3. Solving some simple inequalities

Suppose we want to solve the inequality  $x + 3 > 2$ .

We can solve this by subtracting 3 from both sides:

$$x + 3 > 2$$

$$x > -1$$

So the solution is  $x > -1$ . This means that any value of  $x$  greater than  $-1$  satisfies  $x + 3 > 2$ .

Inequalities can be represented on a number line such as that shown in Figure 1. The solid line shows the range of values that  $x$  can take. We put an open circle at  $-1$  to show that although the solid line goes from  $-1$ ,  $x$  cannot actually equal  $-1$ .

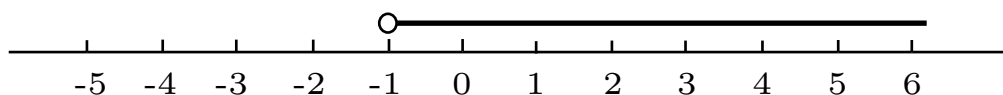


Figure 1. A number line showing  $x > -1$ .

**Example**

Suppose we wish to solve the inequality  $4x + 6 > 3x + 7$ .

First we subtract 6 from both sides to give

$$4x > 3x + 1$$

Now we subtract  $3x$  from both sides:

$$x > 1$$

This is the solution. It can be represented on the number line as shown in Figure 2.

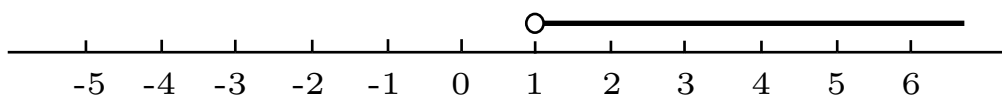


Figure 2. A number line showing  $x > 1$ .

**Example**

Suppose we wish to solve  $3x - 5 \leq 3 - x$ .

We start by adding 5 to both sides:

$$3x \leq 8 - x$$

Then add  $x$  to both sides to give

$$4x \leq 8$$

Finally dividing both sides by 4 gives

$$x \leq 2$$

This is shown on the number line in Figure 3. The closed circle denotes that  $x$  can actually equal 2.

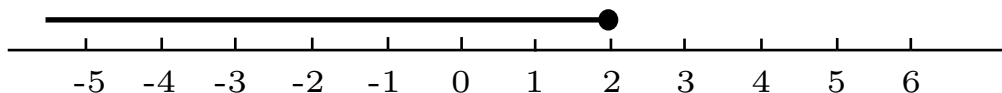


Figure 3. A number line showing  $x \leq 2$ .

**Example**

Suppose we wish to solve the inequality  $-2x > 4$ .

In order to solve this we are going to divide both sides by  $-2$ , and we need to remember that because we are dividing by a negative number we must reverse the inequality.

$$x < -2$$

There is often more than one way to solve an inequality. We are going to solve this one again by using a different method. Starting with  $-2x > 4$  we could add  $2x$  to both sides to give

$$0 > 4 + 2x$$

Then we could subtract 4 from both sides giving

$$-4 > 2x$$

and finally dividing both sides by 2 gives

$$-2 > x$$

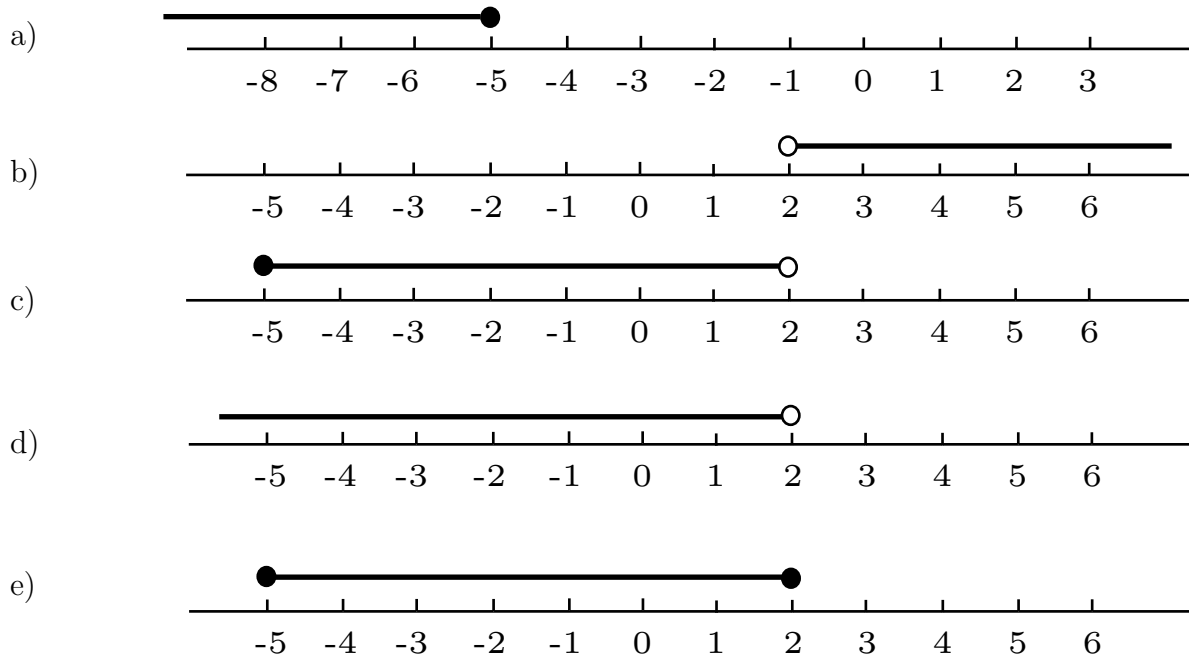
Saying that  $x$  is less than  $-2$  is the same as saying  $-2$  is greater than  $x$ , so both forms are equivalent.

## Exercises 1

1. Draw a number line representation of each of the following inequalities:

- a)  $x > 3$    b)  $x \leq 2$    c)  $-1 < x \leq 2$   
 d)  $x \geq 5$    e)  $4 \leq x < 9$    h)  $-6 < x < 2$

2. Give the inequality which produces the range shown in each of the figures below.



3. Solve the following inequalities

- a)  $3x \leq 9$    b)  $2x + 3 \geq 15$    c)  $-3x < 12$    d)  $2 - 3x < -4$   
 e)  $1 + 5x < 19$    f)  $11 - 2x > 5$    g)  $5x + 3 > 3x + 1$    h)  $12 - 3x < 4x - 2$

## 4. Inequalities used with a modulus symbol

Inequalities often appear in conjunction with the modulus, or absolute value symbol  $| \cdot |$ , for example, in a statement such as

$$|x| < 2$$

Recall that the modulus of a number is simply its magnitude, or absolute value, regardless of its sign. So

$$|2| = 2 \quad \text{and} \quad |-2| = 2$$

Returning to  $|x| < 2$ , if the absolute value of  $x$  is less than 2, then this means that  $x$  must lie between 2 and  $-2$ . We can write this as  $-2 < x < 2$ . This range of values is shown on the number line in Figure 4.

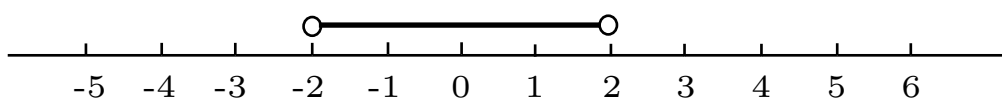


Figure 4. A number line showing  $-2 < x < 2$ .

Observe that  $|x|$  also measures the distance of a point on the number line from the origin. For example, both the points 2 and  $-2$  are distance 2 units from O, and so they have the same absolute value, 2. If we write  $|x| < 2$  we mean all points a distance less than 2 units from O. Clearly these are the points in the interval  $-2 < x < 2$ .

Similarly,  $|x - 4| < 2$  represents all points whose distance from the point 4 is less than 2. These are the points in the interval  $2 < x < 6$ .

### Example

Suppose we wish to solve the inequality  $|x| \geq 5$ .

If  $|x| \geq 5$  this means that the absolute value of  $x$  must be greater than or equal to 5. This means that  $x$  can be greater than or equal to 5, or can be less than or equal to  $-5$ . We write

$$x \leq -5 \quad \text{or} \quad x \geq 5$$

This range of values is shown on the number line in Figure 5.

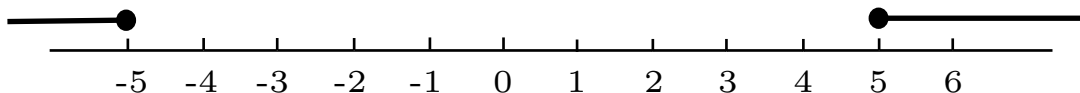


Figure 5. A number line showing  $|x| \geq 5$ .

The next example is more complicated.

### Example

Suppose we wish to solve

$$|x - 4| < 3$$

The modulus sign means that the absolute value of  $x - 4$  is less than 3. This means that

$$-3 < x - 4 < 3$$

This is what is called a double inequality. We must treat it as two separate inequalities.

From the left we get  $-3 < x - 4$  and by adding 4 to both sides we obtain  $1 < x$ .

On the right we have  $x - 4 < 3$ , and by adding 4 to both sides we get  $x < 7$ .

We can write these solutions together as

$$1 < x < 7$$

and this range of values of  $x$  is illustrated on the number line in Figure 6.

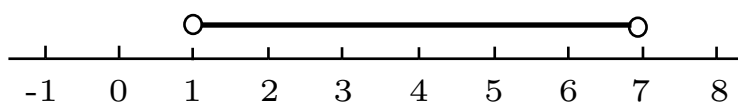


Figure 6. A number line showing  $1 < x < 7$ .

### Example

Suppose we wish to solve  $|5x - 8| \leq 12$ .

This means

$$-12 \leq 5x - 8 \leq 12$$

and again we have a double inequality.

On the left:

$-12 \leq 5x - 8$ . Adding 8 to both sides:  $-4 \leq 5x$ , and dividing by 5 gives,  $-\frac{4}{5} \leq x$ .

On the right:

$5x - 8 \leq 12$ . Adding 8 to both sides:  $5x \leq 20$ . Dividing by 5 gives  $x \leq 4$ .

Putting these results together gives the solution

$$-\frac{4}{5} \leq x \leq 4$$

This range of values is shown on the number line in Figure 7.

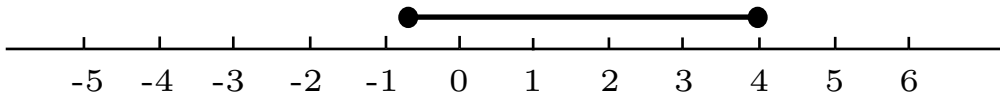


Figure 7. A number line showing the values of  $x$  for which  $|5x - 8| \leq 12$ .

### Exercises 2

Solve the following inequalities

- a)  $|x| \leq 3$     b)  $|x| > 6$     c)  $|x - 4| \leq 3$     d)  $|x - 2| \leq 5$   
e)  $|x + 1| < 3$     f)  $|x + 4| \geq 2$     g)  $|3 - x| > 1$     h)  $|x + 1| \leq 0$

## 5. Using graphs to solve inequalities

Inequalities can be solved very easily using graphs, and if you are in any way unsure about the algebra, it would be a good idea to do a graph to check. Let us see how this works.

### Example

Suppose we wish to solve  $2x + 3 < 0$ .

This inequality could be solved very easily doing algebra, but it makes a good graphical example.

First we sketch a graph of  $y = 2x + 3$  as shown in Figure 8. Note that it is a straight line. It has a slope of 2 and an intercept on the  $y$  axis of 3.

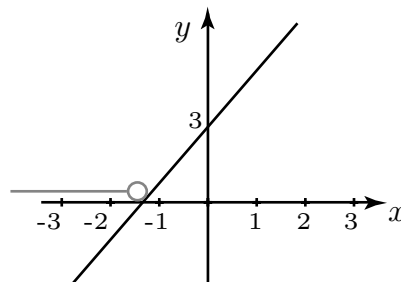


Figure 8. A graph of  $y = 2x + 3$ .

Observe that on the  $x$  axis,  $y = 0$  so that where the graph cuts the  $x$  axis,  $y$  is equal to zero and  $x$  is  $-\frac{3}{2}$ .

Above the  $x$  axis  $y$  is greater than zero.

Below the  $x$  axis  $y$  is less than zero.

Because we are looking for values of  $x$  for which  $2x + 3$  is less than zero, then we look for those points on the graph where  $y$  is less than zero. By inspection we see that this corresponds to values of  $x$  less than  $-1\frac{1}{2}$ . This is the solution of the inequality. We have marked this range on the graph, using the  $x$  axis as the number line.

This technique can also be used with modulus inequalities and here using a graph can be very helpful.

### Example

Suppose we wish to solve the inequality  $|x| - 2 < 0$ .

Again we need to plot the graph of  $y = |x| - 2$ . The graph is shown in Figure 9.

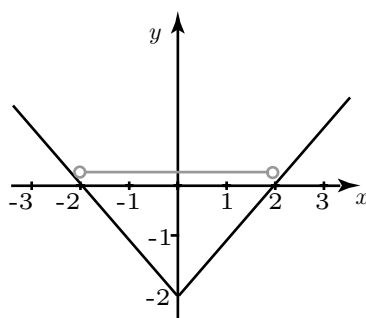


Figure 9. A graph of  $y = |x| - 2$ .

Again we are looking for  $|x| - 2$  to be less than zero, so we are looking for where  $y$  is less than zero. By inspecting the graph we see that this is when  $-2 < x < 2$ . This is the solution of the inequality. This range of values has been marked on the graph using the  $x$  axis as the number line.

### Exercises 3

By drawing appropriate graphs solve the inequalities

- a)  $4x + 3 < 0$       b)  $3 - 2x > 0$       c)  $|x| - 3 > 0$   
d)  $|x - 2| + 4 < 10$     e)  $5x + 1 < 2x + 13$     f)  $x^2 < 3x$

## 6. Quadratic inequalities

Quadratic inequalities need handling with care.

### Example

Suppose we wish to solve  $x^2 - 3x + 2 > 0$ .

The quadratic expression on the left will factorise to give  $(x - 2)(x - 1) > 0$ . If this was a quadratic equation we would simply state  $x - 2 = 0$  and  $x - 1 = 0$  and hence  $x = 2$  and  $x = 1$ . Unfortunately with inequalities the situation is more complicated and we have a bit more work to do.



Whether  $(x - 2)(x - 1)$  is greater than zero or not depends upon the signs of the two factors  $(x - 2)$  and  $(x - 1)$ . We investigate the possibilities using a grid as shown in Figure 10.

On the top line of the grid we have indicated the places where  $(x - 2)(x - 1)$  is equal to zero, that is when  $x$  is 1 or 2.

We write the two factors  $(x - 1)$  and  $(x - 2)$  in the first column on the left. We write their product at the bottom left.

	1	2	
$x - 1$	-	+	+
$x - 2$	-	-	+
$(x - 1)(x - 2)$	+	-	+

Figure 10.

The second column corresponds to where  $x$  is less than 1. When  $x < 1$  both  $x - 1$  and  $x - 2$  will be negative and so we have inserted  $-$  signs to show this. The product  $(x - 1)(x - 2)$  will therefore be positive, and hence the  $+$  sign.

The third column corresponds to where  $x$  is greater than 1 but less than 2. In this interval  $x - 1$  is positive, but  $x - 2$  is negative, and hence the corresponding signs. The product will then be negative.

The fourth column shows what happens when  $x$  is greater than 2. Both factors are positive. Hence their product is positive too.

We are looking for where  $(x - 2)(x - 1) > 0$  and our grid shows us that this is true when  $x < 1$  and when  $x > 2$ . The solution of the inequality is therefore  $x < 1$  or  $x > 2$ . The solution is shown on the number line in Figure 11.

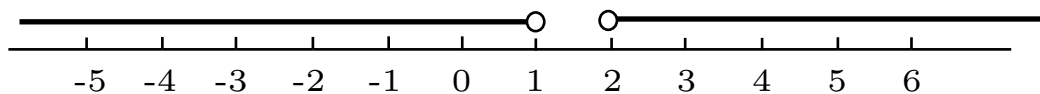


Figure 11.

### Example

Suppose we wish to solve the inequality  $-2x^2 + 5x + 12 \geq 0$ .

It will be easier to deal with this if the coefficient of  $x^2$  is positive rather than negative and so we multiply every term by  $-1$  remembering to reverse the inequality.

The problem then is to solve  $2x^2 - 5x - 12 \leq 0$ .

The quadratic expression can be factorised to give  $(2x + 3)(x - 4) \leq 0$ .

Again we produce a grid. The first factor is zero when  $x = -3/2$ . The second factor is zero

when  $x = 4$ . We write these two numbers on the top row of the grid as shown in Figure 12.

		$-\frac{3}{2}$	4	
$2x+3$	-	+	+	
$x-4$	-	-	+	
$(2x+3)(x-4)$	+	-	+	

Figure 12.

When  $x$  is less than  $-\frac{3}{2}$  both factors are negative and hence their product is positive as indicated. When  $x$  is greater than  $-\frac{3}{2}$  but less than 4,  $2x+3$  is positive, but  $x-4$  is negative. Hence the product of the two factors is negative.

When  $x$  is greater than 4, both factors are positive, and hence their product is positive.

We are looking for where  $2x^2 - 5x - 12 \leq 0$ . From the grid we see that this occurs when

$$-\frac{3}{2} \leq x \leq 4$$

Note that since the quadratic expression is zero at the points  $x = -\frac{3}{2}$  and  $x = 4$  these must be included in the solution. The range of values of  $x$  satisfying the inequality is shown on the number line in Figure 13.

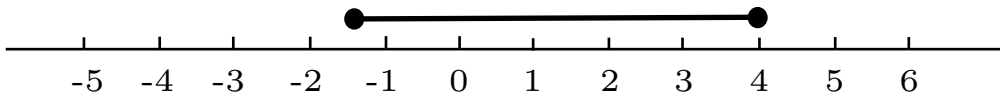


Figure 13.

Quadratic inequalities can also be solved graphically as illustrated in the following example.

**Example**

Suppose we wish to solve  $x^2 - 3x + 2 > 0$ .

We consider the graph of  $y = x^2 - 3x + 2$  which has been drawn in Figure 14. Note that the quadratic expression factorises to give  $y = (x - 1)(x - 2)$  and so the graph crosses the  $x$  axis when  $x = 1$  and when  $x = 2$ . We are looking for where  $x^2 - 3x + 2$  is greater than zero so we look at that part of the graph which is above the  $x$  axis. So the solution is

$$x < 1 \quad \text{or} \quad x > 2$$

We can mark this solution using the  $x$  axis as the number line.

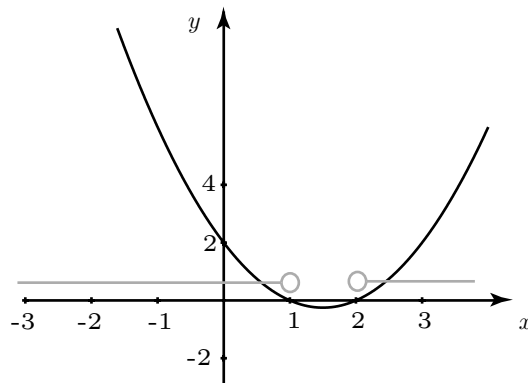


Figure 14.

### Example

Suppose we wish to solve  $x^2 - x - 6 \leq 0$ .

The quadratic expression factorises to  $(x - 3)(x + 2)$  and the graph of  $y = (x - 3)(x + 2)$  is shown in Figure 15.

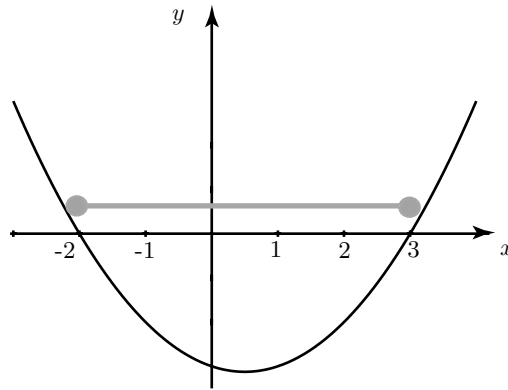


Figure 15.

The graph crosses the  $x$  axis at  $x = -2$  and at  $x = 3$ .

We are looking for where  $x^2 - x - 6$  lies on or below the  $x$  axis. By inspection the solution is

$$-2 \leq x \leq 3$$

Again this solution is indicated on the graph.

### Exercises 4

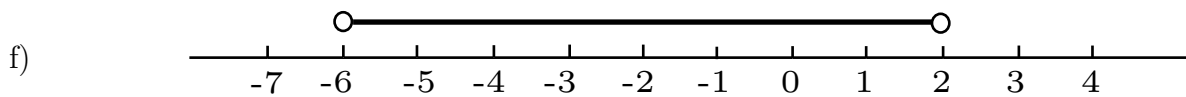
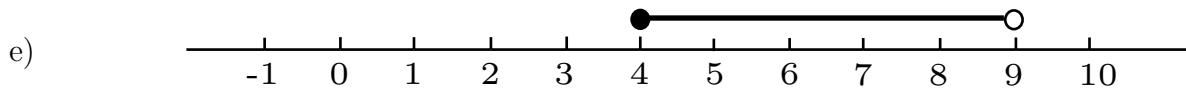
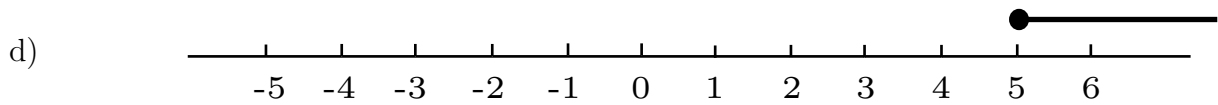
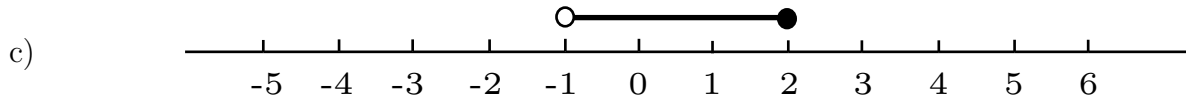
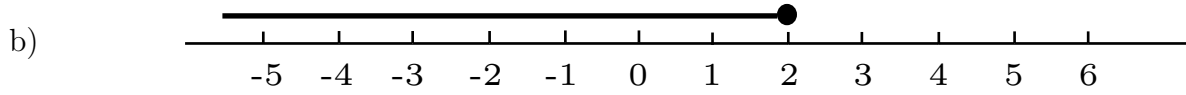
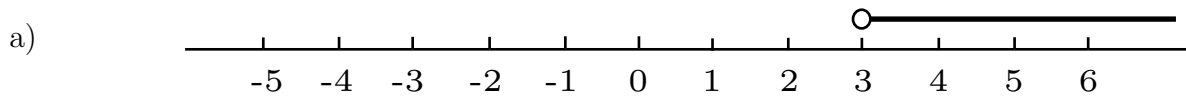
Solve the following quadratic inequalities by using a grid - confirm your answers by sketching the appropriate graph

- a)  $(x - 3)(x + 1) < 0$    b)  $x^2 + 5x + 6 \geq 0$    c)  $(2x - 1)(3x + 4) > 0$   
d)  $10x^2 - 19x + 6 \leq 0$    e)  $5 - 4x - x^2 > 0$    f)  $1 - x - 2x^2 < 0$

## Answers

### Exercise 1

1.



2. a)  $x \leq -5$  b)  $2 < x$  c)  $-5 \leq x < 2$  d)  $2 > x$  e)  $-5 \leq x \leq 2$

3. a)  $x \leq 3$  b)  $x \geq 6$  c)  $x > -4$  d)  $2 < x$  e)  $x < 3.6$  f)  $3 > x$  g)  $x > -1$  h)  $2 < x$

### Exercise 2

a)  $-3 \leq x \leq 3$  b)  $x < -6$  or  $x > 6$  c)  $1 \leq x \leq 7$  d)  $-3 \leq x \leq 7$  e)  $-4 < x < 2$

f)  $x \leq -6$  or  $x \geq -2$  g)  $x < 2$  or  $x > 4$  h)  $x = -1$

### Exercise 3

a)  $x < -4/3$  b)  $x < 3/2$  c)  $x < -3$  or  $x > 3$  d)  $-4 < x < 8$  e)  $x < 4$

f)  $0 < x < 3$

### Exercise 4

a)  $-1 < x < 3$  b)  $x < -3$  or  $x > -2$  c)  $x < -4/3$  or  $x > 1/2$  d)  $2/5 \leq x \leq 3/2$

e)  $-5 < x < 1$  f)  $x < -1$  or  $x > 1/2$